A COMPUTATIONAL PROGRAM FOR AERODYNAMIC GENERALIZED FORCES

by

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ABSTRACT

The aim of the paper is to provide a substantial contribution to the needs of the industry in predicting aeroelastic properties of the aircrafts.

The program presented here is a part of a larger program which includes also consideration of vibration modes, be used in flutter prediction.

Firstly, the Kernel of the well-Known POSSIO-KUSSNER equation is rearranged, so as to isolate singular terms, which are very simple analytically, but must be integrate numerically via special techniques which are described in the paper; the form is chosen in such a way as to integrate function independent of reduced frequency, so that for each of them, only one integration is needed. For regular terms, on the contrary, which exhibit a complicated dependence on reduced frequency, conventional integration techniques are sufficient.

Special attention is devoted to the numerical calculation of Hadamard's principal part.

It is also shown for which reason doublets representing the wing cannot be located on some special points (leading and trailing edges and root and end chords; chords separating elements of different inclinations etc.).

An idea of the technique to be used to remove all difficulties associated with infinitesimal thickness is presented.

The order of magnitude of finite thickness effect is given.

1. — Introduction

As is well known, by the term aerodynamic generalized force (AGF) A_{hk} associated with two modes (h, k) we mean the work performed by the aerodynamic forces on a lifting surface associated with the mode h by the displacements associated with the mode k. As is also known, A_{hk} has a complex character, and, in general, $A_{hk} \neq A_{kh}$.

The knowledge of AGF's is essential in flutter prediction (Fig. 1). Here we have mass and modes

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matrices, which are insensitive to reduced frequency k changes, whereas the AGF matrix does depend on k. Thus we have to face the problem of obtaining zero value of the flutter determinant by an adjustment of k, by means of one of the available trial and error or automatic adjustment methods (e. g. Newton–Raphson).

Hence follows that an efficient method or program for the AGF's computation should satisfy two preliminary basic requirements:

i) be sufficiently flexible to variation of k;

ii) incorporate advanced numerical methods of matrix algebra.

Thus the program presented in this paper must be considered as a phase of the general loop of Fig. 1. Moreover, since calculating the work of given forces by given displacements is a very simple numerical routine, only the problem of determining the distribution of aerodynamic forces on a lifting surface in steady oscillatory motion will be considered in this paper, for the case of subsonic compressible linearized flow.

2. – Theoretical foundation

The theoretical basis for the problem under concern was established many years ago and will not be repeated here [1], [2]. Suffice here to say that the problem reduces to the solution of the integral equation:

(1) $$w(P) = \oint_{\Sigma} G(P, P')\, q(P')\, dP'$$

(for symbols, see the list). As far as the nucleus is concerned, this was also well established several years ago [3], [4]; but in view of satisfying requirement (i) of art. 1, we have split it into six terms, whose analytical expression is given in Appendix I. Inspection of such terms reveals that the first four of them must be calculated once for all, leaving the task of a special computation for each of the last two terms.

Next step is the expression of the unknown q(P) under the form:

(2) $$q(P) = \sum_{o\,m}^{M} \sum_{o\,n}^{N} q_{mn}\, u^m \sqrt{\frac{1-u}{u}}\; \eta^n \sqrt{1-\eta^2}$$

where the nondimensional coordinate u = AB/AC, Fig. 2.

The square roots appearing in (2) stem from the behavior of the flow in the vicinity of the wing edges. Furthermore in Eq. (2), M + 1 is the total number of aeroelastic modes which are taken in the wind direction, and N + 1 is the number of modes in the transverse direction: of course, the higher M and N, the more accurate are the results, and the more time expensive is the program. However, by introducing (2) into (1), and imposing (2) at a total of (N + 1) · (M + 1) control points, we are led to a system of linear equations in the unknowns q_{mn}.

The basic operations for calculating the coefficients are an x–wise and a subsequent y–wise integration.

There is no difficulty in performing such a task as far as G_5 and G_6 are concerned. Likewise, terms G_2 , G_3 , G_4 lead to the contribution described in Appendix II.

A real problem arises in connection with the term G_1 which exhibits a non integrable singularity of order 2.

The root of this lies in the well-known circumstance that the actual form of Eq. (1) would be:

$$\text{(3)} \qquad w\,(P) = \lim_{z \to o} \frac{\partial}{\partial z} \left[\int_\Sigma H\,(P,\,P',\,z)\; q\,(P')\; d\,P \right]$$

where H(P, P′ , z) is the velocity potential due to a pressure doublet acting in the lifting surface plane, and z is the coordinate as measured normally to such surface, of equation z = 0 . Differentiation under sign, wich would be illegal if performed in the conventional way becomes possible if one defines it according to Hadamard's rule.

Thus, with $G\,(P,\,P') = \left(\dfrac{\partial H}{\partial z}\right)_{z=0}$ we obtain Eq. (1).

Let us consider in greater detail what happens as far as the term with G_1 is concerned (the other terms do not present any problem, since they are integrable, and for them Hadamard's rule coincides with the elementary one).

For such term, Eq. (1) reads:

$$\text{(4)} \qquad w\,(x_i\,,\,y_i) = w_{ij} = \sum_{o\,m}^{M} \sum_{o\,n}^{N} q_{mn}\; C_{ij\,,mn}$$

where:

$$\text{(5)} \qquad C_{ij\,,mn} = \,⨎_{1}^{1} \frac{c(\eta)\; d\,\eta\; \eta^n \sqrt{1-\eta^2}}{(y_j-\eta)^2}\; \phi_m\,(x_i\,,y_j\,,\,\eta)$$

and:

$$\text{(6)} \qquad \phi_m(x_i,\,y_j\,,\,\eta) = \int_0^1 \left[1 + \frac{U-u}{\sqrt{(U-u)^2+\epsilon^2}} \right] u^m \sqrt{\frac{1-u}{u}}\; d\,u$$

Numerical evaluation of (5) requires an appropriate integration technique, that was used in the present approach.

We write:

$$\text{(7)} \qquad \frac{\phi_m\,(x_i\,,\,y_j\,,\,\eta)\; c\,(\eta)\sqrt{1-\eta^2}\;\eta^n}{(\eta-y_j)^2} = \psi_{mn}\,(x_i\,,\,y_j\,,\,\eta) =$$

$$= \frac{\psi_m\,(x_i\,,\,y_j\,,\,y_i)}{(\eta-y_j)^2} + \left(\frac{\partial\,\psi_{mn}}{\partial\,\eta}\right)_{\eta=y_j} \frac{1}{(\eta-y_j)} + (\eta-y_j)^2 \left(\frac{\partial^2\,\psi_{mn}}{\partial\,\eta^2}\right)_{\eta=y_j}$$

The expression of the last term is calculated in an exact form, without having recourse to any finite-difference approach; this is possible on account of the circumstance that the integral (6) can be expressed analytically via elliptic integrals.

Subsequent integration of such term is numerically elementary.

As for as the first and second term is concerned, we are confronted with the calculation of the expression:

$$
(8) \quad \begin{cases} PP2 = \psi_m(x_i, y_j, y_j) - \displaystyle\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{d\eta}{(\eta - y_j)^2} \\[2ex] PP1 = \left(\dfrac{\partial \psi_m}{\partial \eta}\right)_{\eta = y_j} \displaystyle\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{d\eta}{(\eta - y_j)} \end{cases}
$$

At this point we must remark that if the wing planform exhibits discontinuities at $\eta = y_j$ in the profile itself or in its derivatives (Fig. 3a, 3b), Eq. (8) would be transformed as follows:

$$
(9) \quad \begin{cases} PP2 \doteq \psi_m^{+} \displaystyle\int_{-1}^{y_j} \frac{d\eta}{(\eta_j - y)^2} + \psi_m^{-} \int_{y_j}^{1} \frac{d\eta}{(\eta - y_j)^2} \\[2ex] PP1 = \left(\dfrac{\partial \psi_m}{\partial \eta}\right)^{+}_{\eta = y_j} \displaystyle\int_{-1}^{y_j} \frac{d\eta}{\eta_j - y} + \left(\dfrac{\partial \psi_m}{\partial \eta}\right)^{-}_{\eta = y_j} \int_{y_j}^{1} \frac{d\eta}{\eta_j - y} \end{cases}
$$

As a matter of fact, according to Eq. (3), the integrals appearing in (9) should be written:

$$
I_2^{\pm} = \pm \lim_{z \to 0} \frac{\partial}{\partial z} \left[z \int_{\mp 1}^{y_j} \frac{d\eta}{z^2 + (y - \eta_j)^2} \right]
$$

$$
I_1^{\pm} = \pm \lim_{z \to 0} \frac{\partial}{\partial z} \left[z \int_{\mp 1}^{y_j} \frac{d\eta}{\sqrt{z^2 + (y - \eta_j)^2}} \right]
$$

leading to:

$$
I_2^{\pm} = \lim_{z \to 0} \frac{\partial}{\partial z} \left[\pm \tan^{-1} \frac{y_j \pm 1}{z} \right] = \mp \frac{1}{y_j \pm 1}
$$

$$
I_1^{\pm} = \lim_{z \to 0} \frac{\partial}{\partial z} \left[\mp z \left(\log \sqrt{1 + u^2} - u \right)_{\frac{y_j \pm 1}{z}}^{o} \right]
$$

It is seen that I_2 is always finite, except for control points lying at the wing tip (which would however be partially compensated by $\sqrt{1 - \eta^2}$), whereas I_1 becomes infinite.

However, if $\left(\frac{\partial \psi_m}{\partial y}\right)^+ = \left(\frac{\partial \psi_m}{\partial y}\right)^-$; i, e. if the planform wing has no points of discontinuities in the edge slopes, the two terms cancel, whereas the reverse is not true.

In other words, Hadamard's rule is not applicable for wings such as that of Fig. 3a, whereas it would be paradoxically valid for Fig. 3b (where, however, other dominant factors would arise; first of all, absolute inadequacy of a linearized theory).

3. — Description of the program

a) *Choise of control points*

According to what has been said above, we must avoid placing control points on the wing tips, and in the discontinuities of the wing. Furthermore, an elementary calculation shows that $\left(\frac{\partial \phi_m}{\partial U}\right)_{\eta=y_j}$ becomes infinite as $u = 0$, $u = 1$ so that also placing control points on the leading and trailing edges would be illegal.

It should also be remembered that all concerns about the singular behavior of the kernel vanish as we are at a reasonable distance from the control point for which the kernel is being evaluated, and where elementary integration techniques can be applied. Therefore, we may divide the wing planform into "branches", Fig. 4, each of them must necessarily consist of a constant slope region; however, for the reason described above, we may also have different branches in a constant slope region. We choose, above all, the values of U for which we want control points, Fig. 5, no restrictions are placed on such choise, except avoidance $u = 0$, and $u = 1$, as said above.

Furthermore we give an approximate value of N, i. e. of control sections, the program automatically distributes the total N in the various branches, with the following rules:

i) avoid wing tips and discontinuities,

ii) if a fraction of N remains the remainder is attributed to the external branches;

iii) the number of elements in every branch must be an even number not smaller than 4 (this is in connection with the requirements for subsequent y–wise integration).

The final number N^* of control sections may be a little greater than the previously prescribed one, in order to avoid unpleasant surprise, it is suggested to have a rough previous estimate.

At the end of the section choice, control points may be distributed as shown in Fig. 6.

A small drawback is associated with the fact that no control points are placed on sections such as MN, PQ, RS of Fig. 6, even if two adjacent branches have the same slopes.

Although this drawback can be easily remedied, the pertinent modification is not included in the program, that would become otherwise too heavy.

b) *Sweep on control points*

For each value of the chosen U's, Fig. 5, and for each branch we sweep on the sections, and for each of them we calculate geometric quantities, and other quantities which solely depend on the control section and on the control point, such as U, ϵ and which must be used in subsequent y–wise integration.

In order to reduce calculation time, different sections are provided for quantities independent of m and n, and for quantities dependent on such exponents.

c) At this point we are ready for calculating the coefficients $C_{ij,mn}$

We must integrate, for the control point under concern, to all the branches, and we must distinguish between control–branches and non-control–branches (relative, of course, to the actual control point).

c1) As far as non control branches are concerned, x–wise integration of all the G_i' s, Appendix II, is elementary, and no logarithmic terms arise. A special subroutine (SUBINX) perform the integration for all the six G_i' s and for all m and n simultaneously. Integration is performed via elementary numerical rules with error control: a special index avoids integration for values of m, n, i for which this is unnecessary.

A similar technique is employed for the y–wise integration.

c2) For the control branch, the functions G_5, G_6 are treated exactly in the same way as it has been described under (c1).

For the other functions a special technique was developed.

Let:

$$c(\eta)\sqrt{1-\eta^2}\cdot\eta^n = F_n(\eta) \tag{10}$$

The integrand of (5) reads:

$$F_n(\eta)\,\phi_m(U(\eta),\epsilon(\eta)) = F_n(\eta)\,\frac{\phi_m(U(\eta),\epsilon(\eta)) - \phi_m(U(\eta),0)}{v^2} + \tag{11}$$

$$+\ F_n(\eta)\,\frac{\phi_m(U(\eta),0) - \phi_m(U(y),0)}{v^2} +$$

$$+\ \phi_m(U(y),0)\,\frac{F_n(\eta) - F_n(y)}{v^2} + \frac{F_n(y)\,Z_m(U(\eta),0)}{v^2}$$

with $v = |y_j - \eta|$.

The first term is evalueted via elliptic integrals, we have:

$$\phi_m(v,\epsilon) = \vartheta^m\, g\, [B_m^{(1)}(v)\cdot v^2 K + B_m^{(2)}(v)\, E + B_m^{(3)}\, v\, \Lambda \tag{12}$$

where the functions $B_m^{(1)}$, $B_m^{(2)}$, $B_m^{(3)}$ are defined in Appendix III, and, of course $\phi_m(U,O)$ has the same expression with $\epsilon = O$.

Furthermore, since:

$$K = K_K\ (v) + K_L\ (v)\ \log\ v$$

$$E = E_K\ (v) + E_L\ (v)\ \log\ v$$

$$\Lambda = \Lambda_K\ (v) + \Lambda_L\ (v)\ \log\ v$$

it is easy to obtain the logarithmic part of (11), to which the special y—wise integration routine is subsequently applied.

The second term is elementary. As far as the third term is concerned we have:

$$\frac{F_n(\eta) - F_n(y)}{v^2} = \frac{F_{np}(y)}{v} + F_{npp}\ (\eta\ ,\ y)$$

(see Appendix IV) of these two factors the first gives rise to PP1, Eq. (7), (8), the second can be integrated by elementary techniques. Finally, the last term of Eq. (11) gives rise to PP2, Eq. (8). By grouping all such terms together we calculate all the expressions (7).

Since also other terms such as G_2 , G_3 can be reduced to a logarithmic and a regular parts, the expressions (12) are used for them too.

Also a logarithmic part is to be extracted from G_4

Once we are ready for the y—wise integration, we also apply here routine integration methods to the regular part and to the logarithmic part.

d) *Error control.*

Error control is done by comparing for every m , n, i the values obtained with the previous ones, if the percent error is greater than a prescribed threshold, the integration step is halved, until the prescribed accuracy is reached.

As for as the regular part is concerned, when the step is halved, the integrals corresponding to the previous approximation do not need be recalculated, since they are the odd steps of the present approximation and the weight is the same for all of them, so that only their sum must be recorded, the same is not true for logarithmic terms, for which weigths change from one step to another and recording all the previously calculated integral would be too heavy from a memory point of view. However, here too a special index for every value of m, n, i (or group of them) automatically blanks out integrals for which the prescribed accuracy has been reached.

e) *Final calculations.* Final calculations involve:

i) inversion of the aeroelastic matrix;

ii) multiplication by modes matrices;

iii) determination of pressures on the lifting surface for each mode;

iv) multiplicazion by the transverse modes matrix and integration to all the lifting surface in order to obtain the A_{hk}'s.

No special comments are deserved by the latest steps, which are based on standard routines.

f) A summary of the routine is described in Fig. 9.

4. – Further improvements

(i) *Thickness effects.* Introduction of finite thickness would imply doubling the unknowns (a doublet plus a source per every point of the lifting surface) and the conditions (upper and lower face of the surface). The effect can be shown to be of the order of the square of thickness chord ratio.

No significant improvements in the solution may be expected, but all singularities and infinites would disappear, although "infinite" quantities would become "very great" quantities, very difficult to be handled numerically. In summary, introduction of thickness effect is considered unnecessary.

(ii) *Wing planform.* The present program is confined to wing with rectilinear edges, simple modifications would allow to consider curved edges too (e. g. circular wings).

(iii) *Control points location.* The difficulties of avoiding sharp variation section may be overcome by the introduction of a small region of finite curvature. Again this drawback is due to doublet scheme coupled with the infinitely small thickness assumption. The pertinent technique may solve an important point of the program.

(iv) *Use of other functions.* Studies on the convergence are being carried out in order to establish which kind of functions, other than the u^m and η^n of Eq. (2) may be more convenient.

A first attempt in substituting orthogonal polynomials to u^m has proven unsatisfactory, on account of too large round–off errors.

5. – Concluding remarks

A program for the calculation of aerodynamic generalized forces has been presented. It is based on the use of oscillating doublet distributions in an inviscid compressible subsonic flow.

Theoretical foundations have been described, location of control points have been discussed, general feature of arithmetic and logic arrangements has been presented. Areas of flutter improvements are indicated.

APPENDIX I

(see Fig. 10)

(I–1)
$$G = G_1 - j\,k\,G_2 + \frac{k^2}{2}\,G_3 + \frac{k^2}{2\beta^2}\,G_4 + G_5 + j\,G_6$$

(I–2)
$$G_1 = \frac{1}{(y-\eta)^2}\left[1 + \frac{x-\xi}{\sqrt{(x-\xi)^2 + \beta^2 (y-\eta)^2}}\right]$$

(I–3)
$$G_2 = \frac{1}{\sqrt{(x-\xi)^2 + \beta^2\,(y-\eta)^2}}$$

(I–4)
$$G_3 = \log\left[\sqrt{(x-\xi)^2 + \beta^2 (y-\eta)^2} - (x-\xi)\right]$$

(I–5)
$$G_4 = G_1 - 1$$

(I–6)
$$G_5 = k^2\left[f_1 (\log \frac{\sqrt{(x-\xi)^2 + \beta^2\,(y-\eta)^2} - (x-\xi)}{2}\,\frac{k}{1-M} +\right.$$

$$\left.\gamma + f_2 + F'\right] - \frac{\ell}{\sqrt{(x-\xi)^2 + \beta^2\,(y-\eta)^2}} +$$

$$+ \frac{k^2}{2}\left[-\frac{M}{\beta^2} + \gamma - \frac{1}{2} + \log\frac{k}{2(1+M)}\right]$$

(I–7)
$$G_6 = k^2\left[\frac{\pi}{2}\,f_1 + F''\right] - \frac{k\,S}{\sqrt{(x-\xi)^2 + \beta^2 (y-\eta)^2}} + \frac{k^2}{2}\,\frac{\pi}{2}$$

where:

$$f_1\,(a) = \frac{1}{2}\sum_{1}^{\infty}{}_m \frac{1}{m!\,(m+1)!}\left(\frac{a}{2}\right)^{2m}$$

$$f_2(a) = -\sum_{1}^{\infty}{}_m \frac{1}{2\,m!\,(m+1)!}\left[\phi\,(m) + \frac{1}{2(m+1)}\right]\left(\frac{a}{2}\right)^{2m}$$

with: $$a = k\,|y-\eta|\ ,\ \phi(m) = \sum_{1}^{m}{}_{r}\,\frac{1}{r}\ ,\ \phi(o) = 0$$

$$\ell\,(B) = \frac{\cos B - 1 - \frac{B^2}{2}}{B}$$

$$S\,(B) = \frac{\sin B - B}{B}$$

with:

$$B = \frac{k}{\beta^2}\left[(x-\xi) - M\sqrt{(x-\xi)^2 + \beta^2 (y-\eta)^2}\right]$$

$$F'\,(a, B) = \sqrt{a^2 + B^2}\ \sum_{1}^{\infty}{}_{m}\ \sum_{1}^{\infty}{}_{k}\ \frac{(-1)^{k+1}}{(2\,m+2)\,!} \cdot \frac{2\,m+1}{2\,m} \cdot \ldots$$

$$\ldots\ \frac{2\,k+1}{2\,k}\ a^{2(m-k)}\ B^{2k-1}$$

$$F''(a\,,\,B) = -\sqrt{a^2 + B^2}\ \sum_{1}^{\infty}{}_{m}\ \sum_{1}^{\infty}{}_{k}\ \frac{(-1)^{k-1}}{(2\,m+1)!}\ \frac{2\,m}{2\,m-1} \ldots$$

$$\ldots\ \frac{2\,k}{2\,k-1}\ a^{2(m-k)}\ B^{2k-2}$$

APPENDIX II

The term G_2 gives to the coefficient $a_{mn}(x, y)$ a contribution given by:

$$a^2_{mn}(x, y) = \int_{-1}^{1} c(\eta)\, d\eta \cdot \eta^n \sqrt{1 - \eta^2} \int_0^1 u^m \sqrt{\frac{1 - u}{u}} \cdot \frac{1}{\sqrt{(u - \eta)^2 + \epsilon^2}}\, du \tag{II–1}$$

Likewiese for the term G_3 we have:

$$a^3_{mn}(x, y) = \int_{-1}^{1} c(\eta) \sqrt{1 - \eta^2} \cdot \eta^n\, d\eta \int_0^1 \frac{U - u}{\sqrt{(U - u)^2 + \epsilon^2}}\, u^m \sqrt{\frac{1 - u}{u}}\, du \tag{II–2}$$

The u–wise integrals in both cases is easily seen to reduce to elliptic integrals (Appendix III). Since, as well known, elliptic integrals in the vicinity of u (modulus) = 1 exhibit a logarithmic singularity, we reduce both integrals to the form $\int a(v)\, dv + \int b(v) \log v\, dv$ where $v = y - \eta$ and $a(v)$, $b(v)$ are regular known functions. Thus the program includes a special subroutine for the calculation of the logarithmic integral which, on the other hand, is known to exist.

APPENDIX III

The following quantities must be computed for (12):

$$U = \frac{x - x'(\eta)}{c(\eta)} \quad ; \quad \epsilon = \frac{\beta\,|y-\eta|}{c(\eta)} ; \quad A = \sqrt{(1-u)^2 + \epsilon^2} \; ;$$

$$B = \sqrt{U^2 + \epsilon^2} \; ; \quad a = \frac{A - B}{A + B} \quad ; \quad k^2 = \frac{1 - (A - B)^2}{4\,A\,B} \quad ;$$

$$k'^2 = 1 - k^2 \; ; \qquad \tau = \frac{k'^2}{k^2} \; ; \; k'_v = \frac{k'}{v}$$

$$\tau_v = \frac{\sqrt{\tau}}{v} ; \; k_v = - \frac{\beta^2\,A\,(1 - 2\,U)/c^2}{(A + B)^2\,[UA + \beta(1-u)]}$$

$$H = 1 - a\,\tau\,(1 - a\tau) \; ; \; g = \frac{1}{\sqrt{A\,B}} \; ; \; \vartheta = - \frac{1}{2(1 + a^2\,\tau)}$$

$$N^{(1)}_{rm} = \sum_{m+1}^{r} j \; N^{(j)}_{rm} \quad ; \quad N^{(2)}_{rm} = \sum_{m+1}^{r} j \; N^{(2j)}_{rm} \quad ; \quad N^{(3)}_{rm} = \sum_{m+1}^{r} j \; N^{(3j)}_{rm}$$

$$m = 0, \ldots, M + 1 \; ; \; r = 0, \ldots, M$$

$$N^{(j)}_{rm} = \sum_{o}^{m-j}{}_q \sum_{o}^{2-1}{}_s \; (-1)^q \binom{m}{j} \binom{m-j}{q} \; H^q \; a^{q+2_r+2_s-(m-2)} C^{(1)}_{rsj} \quad (1 - a^2)$$

$$N^{(2j)}_{rm} = \sum_{o}^{m-j}{}_q \sum_{o}^{2-1}{}_s \; (-1)^q \binom{m}{j} \binom{m-j}{q} \; H^q \; a^{q+2_r+2_s-(m-2)} \; C^{(2)}_{rsj} \; \frac{1}{k^2}$$

$$N^{(3j)}_{rm} = \sum_{o}^{m-j}{}_q \sum_{o}^{r-1}{}_s \; (-1)^q \binom{n}{j} \binom{m-j}{q} \; H^q \; a^{q+2_r+2_s-(m-1)} \; C^{(3)}_{rsj} \; \frac{\pi}{2} \, \frac{1}{\sqrt{1-a^2}} \; \frac{1}{\sqrt{a^2\,k^2 + k^2}}$$

The coefficients $C^{(i)}_{rsj}$ must be computed by means of the recursion formula:

$$C^{(j)}_{hk} = \delta \binom{j-1}{h} \binom{j-2}{k-1} \; (-1)^{j-k-1} - \Big[(4 - 2\,\eta_j) \; C^{(j-1)}_{hk} \; +$$

$$+ (-6 + 6\eta_j)\left(C^{(j-2)}_{h-1,k-1} - C^{(j-2)}_{h-1,k} + C^{(j-2)}_{h,k-1} - C^{(j-1)}_{h,k}\right) +$$

$$+ (4 - 6\eta_j)\left(C^{(j-3)}_{h-2,k-2} - 2C^{(j-3)}_{h-2,k-1} + C^{(j-3)}_{h-2,k} + 2C^{(j-3)}_{h-1,k-2}\right.$$

$$\left. - 4C^{(j-3)}_{h-1,k-1} + 2C^{(j-3)}_{h-1,k} + C^{(j-3)}_{h,k-2} - 2C^{(j-3)}_{h,k-1} + C^{(j-3)}_{h,k}\right) +$$

$$+ (-1 + 2\eta_j)\left(C^{(j-4)}_{h-3,k-3} - 3C^{(j-4)}_{h-3,k-2} + 3C^{(j-4)}_{h-3,k-1} - C^{(j-4)}_{h-3,k} +\right.$$

$$3C^{(j-4)}_{h-2,K-3} - 9C^{(j-4)}_{h-2,k-2} + 9C^{(j-4)}_{h-2,k-1} - 3C^{(j-4)}_{h-2,k} +$$

$$3C^{(j-4)}_{h-1,k-3} - 9C^{(j-4)}_{h-1,k-2} + 9C^{(j-4)}_{h-1,k-1} - 3C^{(j-4)}_{h-1,k} +$$

$$\left. + C^{(j-4)}_{h,k-3} - 3C^{(j-4)}_{h,k-2} + 3C^{(j-4)}_{h,k-1} - C^{(j-4)}_{h,k}\right) +$$

$$+ (2 - \eta_j)\left[C^{(j-1)}_{h,k-1} - C^{(j-1)}_{h-1,k}\right] +$$

$$+ (-1 + \eta_j)\left[C^{(j-2)}_{h-1,k-2} - C^{(j-2)}_{h-1,k-1} + C^{(j-2)}_{h,k-2} - C^{(j-2)}_{h,k-1}\right.$$

$$\left. - C^{(j-2)}_{h-2,k-1} + C^{(j-2)}_{h-2,k} - C^{(j-2)}_{h-1,k-1} + C^{(j-2)}_{h,k}\right]$$

where $\delta = 0$ for $\ell = 2, 3$ $\delta = 1$ for $\ell = 1$.

The starting values one:

$C^{(1)}_{-1,0,0} = 0$	$C^{(2)}_{-1,0,0} = 0$	$C^{(3)}_{-1,0,0} = 0$
$C^{(1)}_{0,0,0} = 0$	$C^{(2)}_{0,0,0} = 0$	$C^{(3)}_{0,0,0} = 0$
$C^{(1)}_{1,0,0} = 1$	$C^{(2)}_{1,0,0} = 0$	$C^{(3)}_{1,0,0} = 1$
$C^{(1)}_{2,0,0} = -3$	$C^{(2)}_{2,0,0} = -1$	$C^{(3)}_{2,0,0} = -2$
$C^{(1)}_{2,0,1} = 2$	$C^{(2)}_{2,0,1} = 0$	$C^{(3)}_{2,0,1} = 1$
$C^{(1)}_{2,1,0} = -2$	$C_{2,1,0} = 0$	$C^{(3)}_{2,1,0} = -1$
$C^{(1)}_{2,1,1} = 1$	$C_{2,1,1} = 0$	$C^{(3)}_{2,1,1} = 0$

$$M_{rm}^{(\ell)} = \delta_o \; u N_{rm}^{(\ell)} - \delta_1 \; \vartheta (1+u) \; N_{r,m+1}^{(\ell)} + \phi^2 \; N_{r,m+2}^{(\ell)}$$

$$B_m^{(\ell)} = \sum_{o}^{m+1}{}_r \; \tau^r \; M_{rm}^{(\ell)} - \mu_1^{(\ell)} + \mu_o^{(\ell)}$$

where:

$$\mu_1^{(\ell)} = 1 \quad \text{for} \quad \ell = 2{,}3 \quad ; \quad \mu_1^{(\ell)} = \tau_v^2 \quad \text{for} \quad \ell = \ell$$

$$\mu^{(\ell)} = 0 \quad \text{for} \quad \ell = 2{,}3 \quad ; \quad \mu_o^{(\ell)} = k_v \quad \text{for} \quad \ell = \ell$$

$$\phi_m = g \; \vartheta^m \; [B_m^{(1)} \circ v^2 \; K + B_m^{(2)} \; E + B_m^{(3)} \; \Lambda]$$

APPENDIX IV

Let:

$$\text{(IV-1)} \qquad \left\{ \begin{aligned} A &= \sqrt{1-\eta^2} \\ B &= c\,(\eta) \\ C &= \eta^n \end{aligned} \right.$$

we have:

$$A(\eta) = \sqrt{1-\eta^2} = \sqrt{1-y^2} + A'\ v \qquad (v = \eta - y)$$

whence:

$$A' = \frac{\sqrt{1-\eta^2} - \sqrt{1-y^2}}{\eta - y} = -\ \frac{\eta + y}{\sqrt{1-\eta^2} + \sqrt{1-y^2}}$$

Again:

$$A' = -\ \frac{y}{\sqrt{1-y^2}} + A''\,(\eta - y)^2$$

with simple algebraic manipulation we find:

$$\text{(IV-2)} \qquad A'' = A''(\eta, y) = -\ \frac{y + \eta}{\sqrt{1-y^2}\,(\sqrt{1-y^2} + \sqrt{1-\eta^2})\,(y\sqrt{1-\eta^2} + \eta\sqrt{1-y^2})}$$

So, in exact form:

$$\text{(IV-3)} \qquad A(\eta) = A(y) + A_p(y)\,v + A_{pp}(\eta, y)\,v^2$$

where: $A_p = -\ \dfrac{y}{\sqrt{1-y^2}}$, and: $A_{pp} = A''$.

By the same ..... ;

$$B = B\,(y) + B_p\,(y)\,v \qquad \text{(IV-4)}$$

$$C = C\,(y) + C_p\,(y)\,v + C_{pp}\,(\eta\,,\,y)\,v^2$$

where:

$$B\,(y) = C\,(y) \quad ; \quad B_p = \frac{d\,c}{d\,\eta}$$

$$C\,(y) = y^n \quad ; \quad C_p\,(y) = n\,y^{n-1} \quad ; \quad C_{pp}\,(\eta\,,\,y) = \sum_{o}^{n}{}_{r} \left(\frac{n}{r}\right) \epsilon^{n-2}\ y^{n-2}$$

It should be noticed that Eqs (IV–3), (IV–4) can be considered exact expressions of the various quantities stopped at second order of v, and all the functions appearing in them can be computed without having recourse to finite differences, but through finite arithmetic operations only.

This procedure, that has been extensively used throughout the problem to all terms appearing in (11) completely avoids significant round–off errors.

By combining Eqs. (IV–3) , (IV–4) one finally gets:

$$\frac{F\,(\eta) - F\,(y)}{v^2} = \frac{F_p}{v} + F_{pp}\,(\eta\,,\,y)$$

where $F = A\ B\ C$ and:

$$F_p = A\,(y)\ B(y)\ C_p(y) + A_p\,(y)\ B\,(y)\ C\,(y) + A\,(y)\ B_p\,(y)\ C(y)$$

whereas the lenghly expression for $F_{pp}\,(\eta\,,y)$ is left to the reader.

NOMENCLATURE

c	=	chord
k	=	reduced frequency
$q(P)$	=	doublet distribution
q_{mn}	=	coefficients of expansion of q
u	=	nondimensional coordinate, Fig. 2
v	=	$\mid y_j - \eta \mid$
w	=	dynamic aeroelastic local slope
x	=	chordwise coordinate, Fig. 2
y	=	spanwise coordinate, Fig. 2
z	=	coordinate normal to lifting surface plane
A_{hk}	=	aerodynamic generalized force
$B_m^{(1)}, B_m^{(2)}, B_m^{(3)}$	=	see Appendix III
$C_{ij,mn}$	=	see (5)
E	=	elliptic integral of second Kind
E_k, E_L	=	see page 7.
F_n	=	see (10)
F_{np}, F_{npp}	=	see Appendix IV and page
$G(P, P')$	=	kernel of integral equation
G_j $(j = 1,\ldots, 6)$	=	terms of G
$H(P, P', z)$	=	velocity potential
K	=	elliptic integral of first Kind
K_k, K_L	=	see page 7.
$I_1^{\pm}$	=	see page 4.
$I_2^{\pm}$	=	see page 4.
M	=	number of chordwise aeroelastic modes
M_o	=	Mach number
N	=	number of spanwise aeroelastic modes
N^*	=	number of control sections
PP1	=	see (8)
PP2	=	see (9)

$U(\eta)$	=	$\dfrac{x - x'(\eta)}{c(\eta)}$
β	=	$\sqrt{1 - M_o^2}$
$\epsilon(\eta)$	=	$\dfrac{\beta \lvert y - \eta \rvert}{c(\eta)}$
η	=	nondimensional spanwise abscissa
Λ	=	Heuman's Lambda Function
Λ_k, Λ_L	=	see page 7.
Σ	=	integration surface
ϕ_m	=	see (6)
ψ_m	=	see (7)

SUBSCRIPTS

i	=	chordwise index of control points
j	=	spanwise index of control points
h, k	=	mode index

other subscripts are defined above

SPECIAL SYMBOLS

$\displaystyle ⨍$ = principal part of an integral (Hadamard's rule)

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NOTICE

At the moment of the presentation of the paper, the program was completed, but not wholly tested, on account of difficulties in availability of computer time. It is expected that the final list, including numerical examples will be ready by December 1976.

It will send – post prepaid – to everyone who makes a request to:

Istituto di Tecnologia Aerospaziale
Università di Roma
Via Eudossiana 18 , ROME, ITALY

Attn. Prof. Paolo Santini.

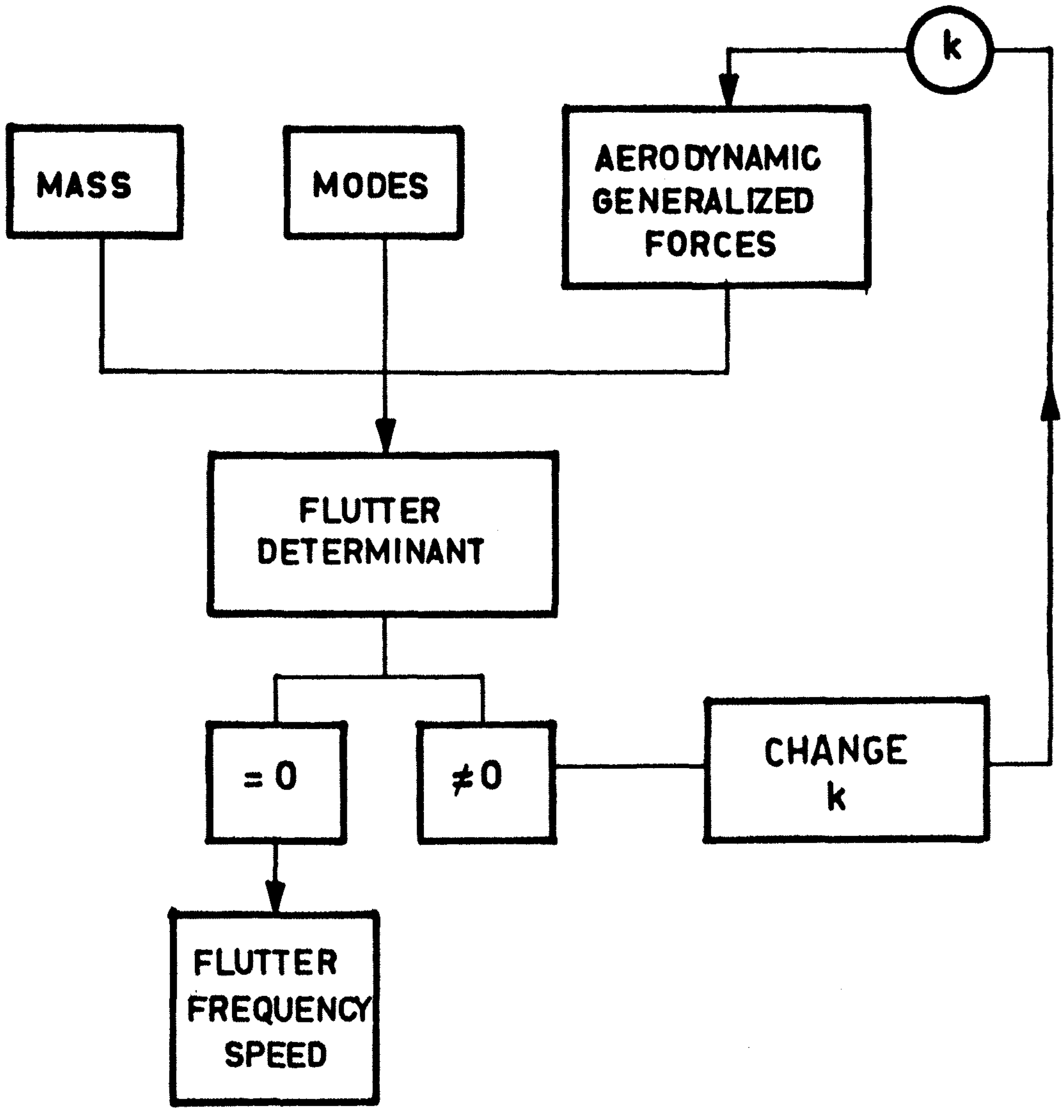
k
MASS
MODES
AERODYNAMIC
GENERALIZED
FORCES
FLUTTER
DETERMINANT
= 0
≠ 0
CHANGE
k
FLUTTER
FREQUENCY
SPEED

FIG. 1

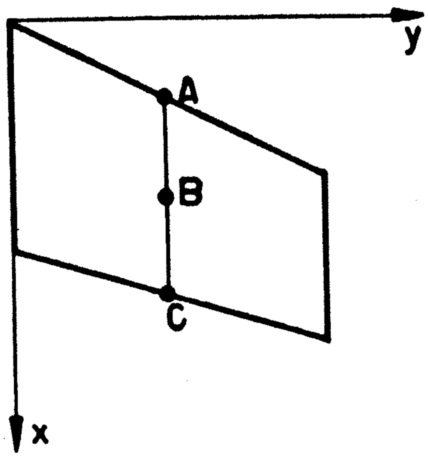

FIG. 2

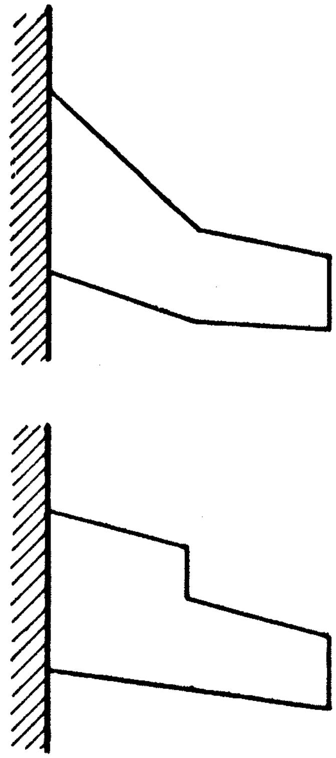

FIG. 3a

FIG. 3b

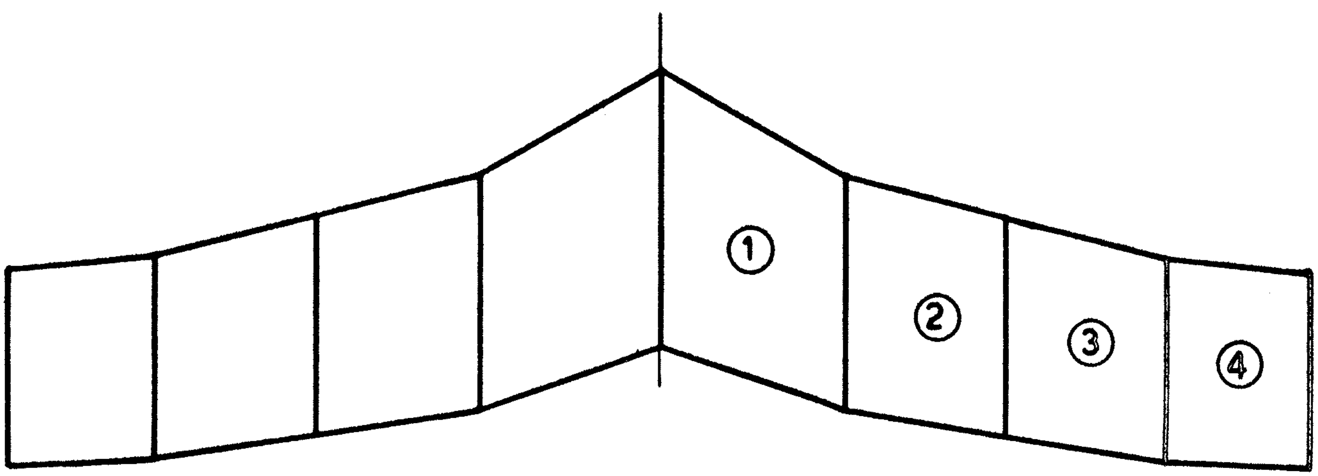

FIG. 4

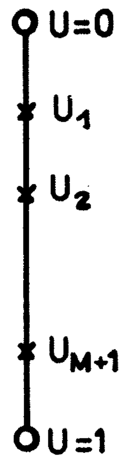

FIG. 5

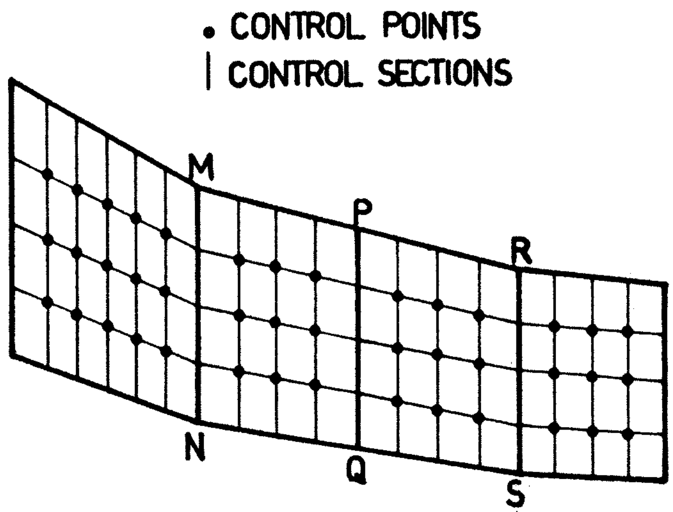

FIG. 6

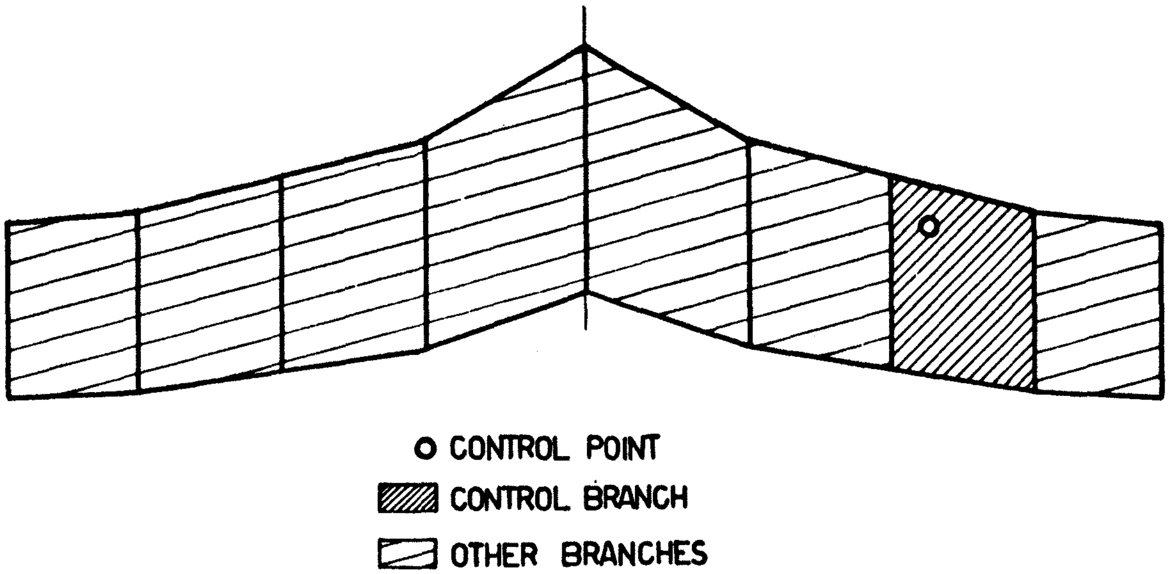

FIG. 7

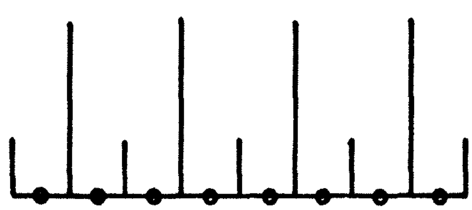

FIG. 8

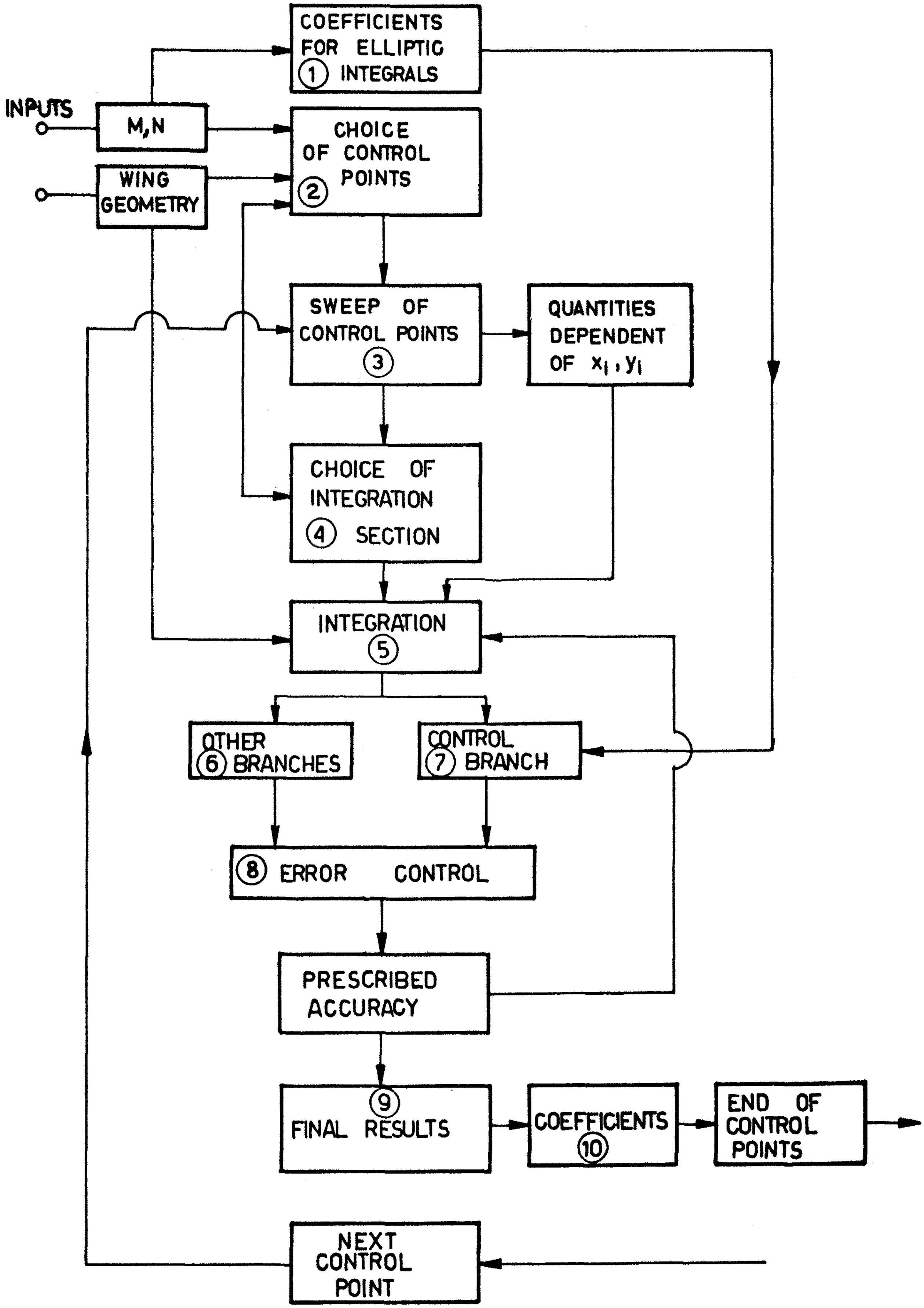
INPUTS
M,N
WING GEOMETRY
COEFFICIENTS FOR ELLIPTIC (1) INTEGRALS
CHOICE OF CONTROL (2) POINTS
SWEEP OF CONTROL POINTS (3)
QUANTITIES DEPENDENT OF x_i, y_i
CHOICE OF INTEGRATION (4) SECTION
INTEGRATION (5)
OTHER (6) BRANCHES
CONTROL (7) BRANCH
(8) ERROR CONTROL
PRESCRIBED ACCURACY
(9) FINAL RESULTS
COEFFICIENTS (10)
END OF CONTROL POINTS
NEXT CONTROL POINT

FIG. 9

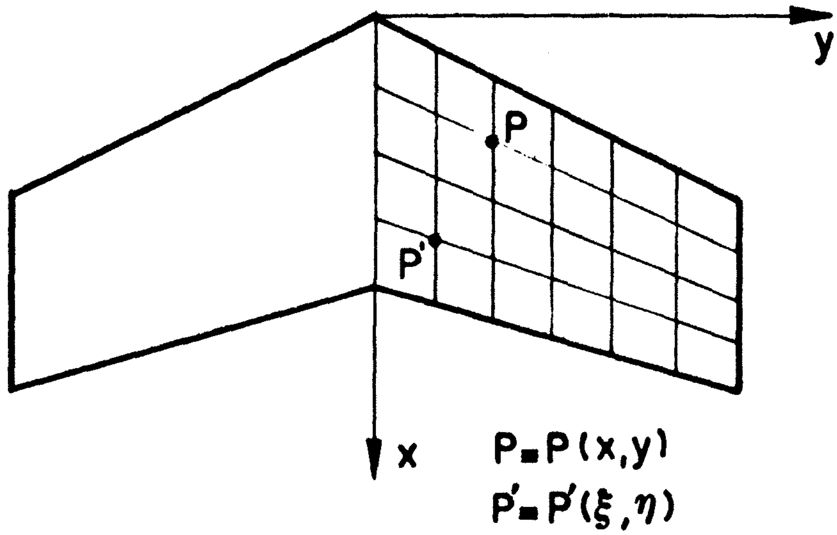

FIG. 10